The Farthest Color Voronoi Diagram and Related Problems

Extended Abstract

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Suppose there are k types of facilities, e. g. schools, post offices, supermarkets, modeled by n colored points in the plane, each type by its own color. One basic goal in choosing a residence location is in having at least one representative of each facility type in the neighborhood. In this paper we provide algorithms that may help to achieve this goal for various specifications of the term "neighborhood". Several problems on multicolored point sets have been previously considered, such as the *bichromatic closest pair*, see e. g. Preparata and Shamos [14, Section 5.7], Agarwal et al. [1], and Graf and Hinrichs [8], the *group Steiner tree*, see Mitchell [11, Section 7.1], or the *chromatic nearest neighbor search*, see Mount et al. [12].

Let us call a set *color-spanning* if it contains at least one point of each color. A natural approach to the above location problem is to ask for the center of the smallest color-spanning circle. For $k = n$ this amounts to finding the smallest circle enclosing n given points. This problem can be solved in time $O(n \log n)$ by means of the farthest site Voronoi diagram [3], in time $O(n)$ using Megiddo's linear programming method [10], or in randomized time $O(n)$ by Welzl's minidisk algorithm [16]. A special case is $k = 2$, then the solution is given by the bichromatic closest pair, see above. For $2 < k < n$ one can solve the problem as follows.

Let us generalize Voronoi diagrams in the following way. If p denotes a site of color c, we put all points of the plane in the region of p for which c is the farthest color, and p the nearest c-colored site, i. e., z belongs to the

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region of p iff the closed circle centered at z that passes through p contains at least one point of each color, but no point of color c is contained in its interior. We call the resulting planar subdivision the *Farthest Color Voronoi Diagram*, *FCVD* for short, see Figure 1 for an example. For $k = n$ the *FCVD*

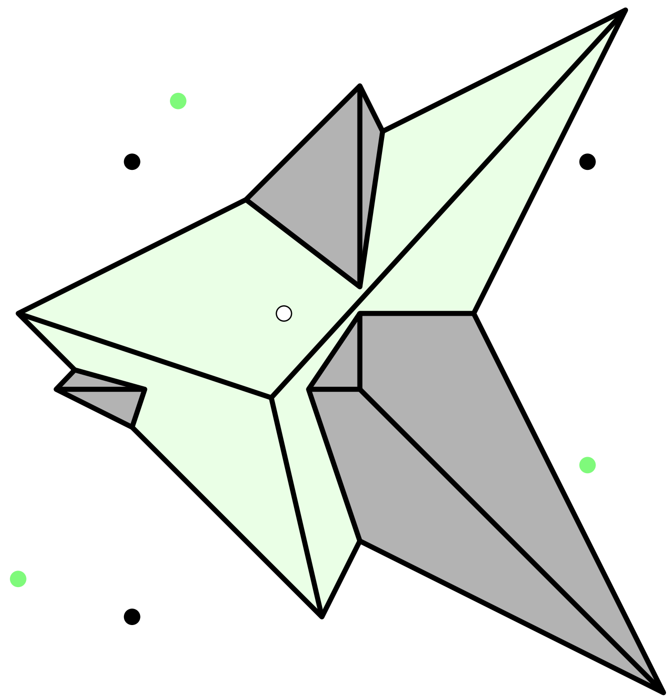

Figure 1: The farthest color Voronoi diagram for three colors and seven sites. The doubly connected unbounded area belongs to the ○ site in the middle while the dark (■) and light (■) shaded areas belong to the closest ● resp. ● sites.

becomes the farthest site Voronoi diagram, while for $k = 1$ we have the usual Voronoi diagram of n points. The *FCVD* is identical to the orthogonal projection on the plane of the upper hull of the one-colored *Voronoi surfaces* as described by Huttenlocher et al. [9] and Sharir and Aggarwal [15, Section 8.7], they use it to obtain the minimum Hausdorff distance between two point sets. For its computation, they give an $O(kn \log n)$ time algorithm.

Once the *FCVD* is given, one can, in time $O(nk)$, determine the smallest color-spanning circle: its center is either a 3-colored vertex, or the midpoint of a 2-colored edge. We complement these results by proving that the *FCVD* has complexity $\Theta(nk)$ if $k \leq \frac{n}{2}$.

Clearly, the *FCVD* can be generalized to other distance measures, e. g. to convex distance functions. This would allow us to compute the smallest color-spanning rectangle of fixed orientation and fixed aspect ratio by the same method as before. An interesting problem arises if the aspect ratio is not specified.

So let us assume that, in the residence location problem, we want to determine a color-spanning axis parallel rectangle of minimum area or minimum perimeter. In principle, we could use a 3-dimensional *FCVD* of colored vertical lines for this problem, where a horizontal cross-section at height z equals the *FCVD* for aspect ratio z. However, we present a more direct approach which has some similarities to the computation of the smallest rectangle or polygon containing at least k of n points, see the articles by Agarwal et al. [2], Datta et al. [4], Dobkin et al. [5], or Eppstein and Erickson [6], for example. Our algorithm constructs the smallest color-spanning rectangle in time $O(n(n-k)\log^2 k)$, using a technique by Overmars and van Leeuwen [13] for dynamically maintainig maximal elements. We also give a simple algorithm whose $O(n(n-k)^2)$ running time is advantageous for large values of k.

Finally, we mention the computation of the narrowest color-spanning strip of arbitrary orientation in time $O(n^2\alpha(k)\log k)$. While a lower bound for the smallest color-spanning rectangle seems unclear, we can show that the narrowest color-spanning strip problem is 3SUM-hard in the sense of Gajentaan and Overmars [7].

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