The Polygon Exploration Problem II: The Angle Hull

Frank Hoffmann* Christian Icking** Rolf Klein**
Klaus Kriegel*

Abstract

Let D be a connected region inside a simple polygon, P. We define the *angle hull* of D, $\mathcal{AH}(D)$, to be the set of all points in P that can see two points of D at a right angle. We show that the perimeter of $\mathcal{AH}(D)$ cannot exceed in length the perimeter of D by more than a factor of 2. This upper bound is tight. Our result can be generalized to angles different from 90°, and to settings where region D is surrounded by obstacles other than a simple polygon.

Key words: Angle hull, computational geometry, convex hull, curve length, motion planning, polygon.

1 Introduction

In on-line navigation algorithms for autonomous robots, analyzing the length of the robot's path is often a complicated issue. Sometimes, only the discovery of certain structural properties has lead to a reasonably sharp analysis; see [1, 4, 5, 6] and Rote [7].

In this paper we provide a new result of this type. It is crucial in analyzing the performance of an on-line strategy for exploring unknown simple polygons presented in Part I of this paper [3]. Yet, the hull construction presented here is independent of the exploration problem, and it seems to be interesting in its own right.

Let D be a bounded, connected region in the plane. For convenience, we shall assume that D is a simple polygon; but our results can easily be generalized to

*Freie Universität Berlin, Institut für Informatik, D-14195 Berlin.
**FernUniversität Hagen, Praktische Informatik VI, D-58084 Hagen.

This work was supported by the Deutsche Forschungsgemeinschaft, grant Kl 655/8-3.

A preliminary version [2] of this paper has appeared in the proceedings of SWAT '98.

curved objects by approximation. Now suppose that a photographer wants to take a picture of D that shows as large a portion of D as possible, but neither white space nor obstacle walls. The photographer is using a fixed angle lens. For now, we assume that the angle equals 90°; later, in Section 5, we show how to generalize to arbitrary angles.

Before taking the picture, the diligent photographer walks around D and inspects all possible viewpoints. We are interested in comparing the length of the photographer's path to the "extension" of the object, D.

In the simple outdoor setting there are no obstacles that can obstruct the photographer's view of D; this situation is depicted in Figure 1. At each point

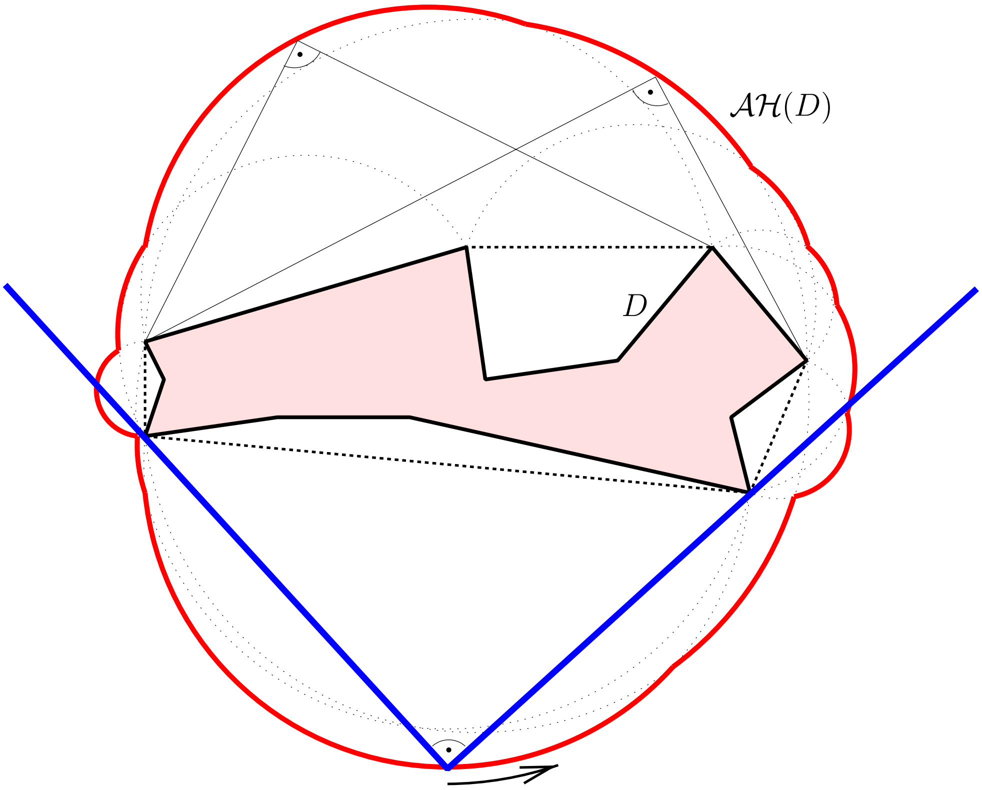

Figure 1: Drawing the angle hull $\mathcal{AH}(D)$ of a region D.

of the path, the two sides of the lens' angle touch the boundary of D from the outside, in general at a single vertex each. Only such vertices can be touched that are situated on the convex hull of D. Consequently, the photographer's path depends on the convex hull, $\mathcal{CH}(D)$, of D, rather than on D itself.

While the right angle is touching two vertices, v and w, of D, its apex describes a circular arc spanned[1] by v and w, as follows from Thales' theorem.

[1]The smallest circle containing two points, v and w, is called the circle *spanned* by v and w.

All points enclosed by the photographer's path, and no other, can see two points of D at a 90° angle; we call this point set the *angle hull* of D and denote it by $\mathcal{AH}(D)$.

More complicated is the indoor setting where D is contained in a simple polygon P whose edges give rise to visibility constraints. The photographer does not want any wall segments to appear in the picture; thus, the viewing angle can now be constrained in different ways: Either side may touch a convex vertex of D that is included in the angle, as before; or it may touch a reflex vertex[2] of P that is excluded; see Figure 2. Any combination of these cases is possible.

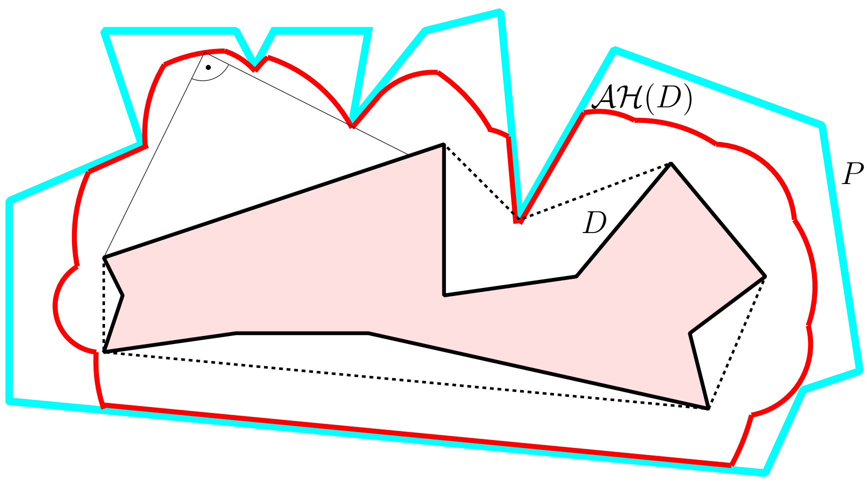

Figure 2: The angle hull $\mathcal{AH}(D)$ inside a polygon P.

As a consequence, the photographer's path contains circular arcs spanned by vertices of D and of P; in addition, it may contain segments of edges of P that prevent the photographer from stepping back far enough; see Figure 2.

Formally, we define the angle hull, $\mathcal{AH}(D)$, of D with respect to P to be the set of all points of P that can see two points of D at a right angle. Its boundary equals the photographer's path.

In the indoor setting, the angle hull $\mathcal{AH}(D)$ depends only on the relative convex hull,[3] $\mathcal{RCH}(D)$, of D. We are interested in an upper bound to the length of the photographer's path around D in terms of the length of the perimeter of $\mathcal{RCH}(D)$.

In Section 2 we show that, in the indoor setting, the angle hull may have twice the perimeter of D, in the limit. Then, in Section 3, we prove that this is the worst that can happen: the angle hull's perimeter cannot exceed twice

[2]A vertex of a polygon is called *reflex* if its internal angle exceeds 180°, as opposed to convex.

[3]The *relative convex hull* of a subset D of a polygon P, $\mathcal{RCH}(D)$, is the smallest subset of P that contains D and, for any two points of D, the shortest path in P connecting them; see the dotted line in Figure 2.

the perimeter of the relative convex hull. This remains true if we have several obstacles instead of one surrounding polygon. For the outdoor setting an even smaller bound of $\pi/2$, which is also tight, can be established in Section 4. In Section 5 we show how to generalize these bounds to angles different from 90°.

As a spin-off result we obtain a tight upper bound to the length of a graph, G, of a non-negative real valued function f satisfying $f(0) = 0, f(1) = 0$, in terms of the minimum angle with apex on G whose sides intersect with the X-axis but not with G.

Finally, we mention some open problems concerning the functorial properties of the angle hull constructor.

2 The lower bound

We start with the proof that the angle hull of a set D contained in a polygon P can be twice as long as the perimeter of D. Our construction is rather simple, D is a line segment and P is a jagged halfcircle.

Lemma 1 *Let $\varepsilon > 0$. There is a polygon, P, and a relatively convex region, D, inside P, for which the boundary of the angle hull $\mathcal{AH}(D)$ with respect to P is longer than $2 - \varepsilon$ times the boundary of D.*

Proof. As our region D, we take a horizontal line segment of length 1. Let p_0, $\ldots$, p_n be equidistant points on the halfcircle spanned by D, where p_0 and p_n are the endpoints of D; see Figure 3. From each point p_i we draw the right angle to the endpoints of D. Let P be the concatenation of the upper envelope of these angles and its reflection at D. Then we have $P = \mathcal{AH}(D)$ by construction. Let us analyze the upper envelope.

We will show that the length of the jagged line from p_0 to p_n is less than 2, but comes arbitrarily close to 2, as n increases. Let q_i be the intersection of the segments $p_0\, p_{i+1}$ and $p_i\, p_n$. If we rotate, for all i, the ascending segments $q_i\, p_{i+1}$ about p_0 onto D, see the dotted arcs in Figure 3, these segments cover disjoint pieces of D, so the total length of all ascending segments is always less than 1. By symmetry, the same bound holds for the descending segments. It remains to show that the ascending length can come arbitrarily close to 1.

Consider the triangle $p_i\, q_i\, p_i'$, where p_i' is the orthogonal projection of p_i onto $p_0\, q_i$. Point p_0 is closer to p_i' than to p_i, so for the distances from p_0 to p_i and to q_i we have

$$|p_0\, q_i| - |p_0\, p_i| \leq |p_0\, q_i| - |p_0\, p_i'| = |p_i'\, q_i| = |p_i\, q_i| \sin \frac{\pi}{2n}.$$

The total length of all ascending segments is therefore 1 minus the following rest.

$$\sum_i (|p_0\, q_i| - |p_0\, p_i|) \leq \sin \frac{\pi}{2n} \sum_i |p_i\, q_i| \leq \sin \frac{\pi}{2n}$$

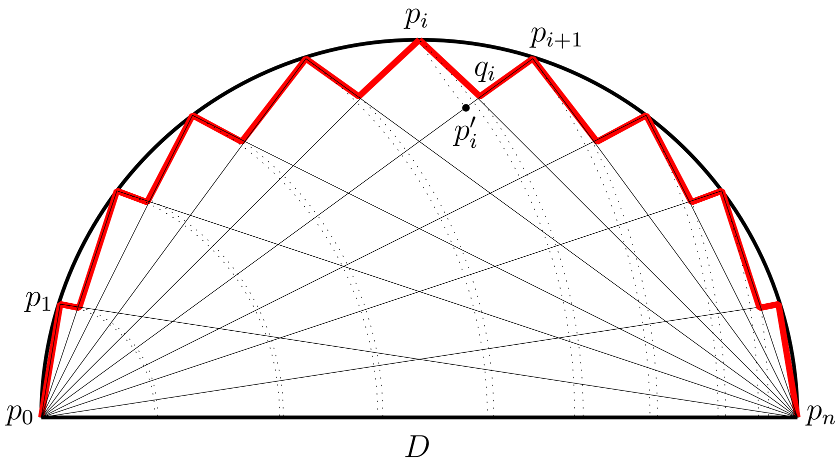

Figure 3: The boundary of the upper envelope of the right angles is less than $2|D|$.

For $n \to \infty$, this tends to 0. The last inequality holds because $\sum_i |p_i\, q_i| \leq 1$ is the length of all descending segments. □

The proof also works for non-equidistant points as long as the maximum distance between subsequent points tends to 0. We are obliged to R. Seidel [8] for this elegant proof of Lemma 1.

3 The upper bound

Interestingly, the same jagged lines as used in the proof of Lemma 1 are also very useful in the proof of the upper bound. For any circular arc C we can construct a jagged line by distributing auxiliary points along C and by taking the upper envelope of the right angles at these points whose sides pass through the two spanning vertices of C; see Figure 4. We denote with *jagged length*, $J(C)$, of C the limit of the lengths of these jagged lines as the maximum distance between subsequent points tends to 0. This limit is well-defined, i. e., it does not depend on how the points are chosen. In the proof of Lemma 1 we have already seen how to determine this length by separately estimating the lengths of the ascending and descending segments. For the jagged length of a circular arc with diameter 1 from angle α to angle β, see Figure 4, we obtain analogously to the proof of Lemma 1

$$J(C) = \sin\beta - \sin\alpha - \cos\beta + \cos\alpha$$

which can also be written as

$$J(C) = \int_{\alpha}^{\beta} (\cos\gamma + \sin\gamma)\, d\gamma\,.$$

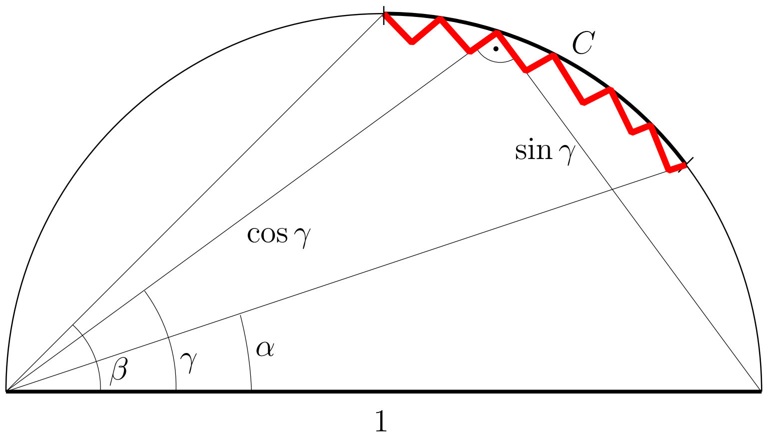

Figure 4: Analyzing the jagged length, $J(C)$, of a circular arc C.

Lemma 2 *The jagged length of an arc is always greater than the arc length itself.*

Proof. Consider a circle with diameter d and a circular arc a on its boundary. Two lines from an arbitrary point on the boundary through the endpoints of the arc always intersect in the same angle ϕ, by the generalized Thales' theorem. For the length of a, we have $|a| = \phi d$; see Figure 5.

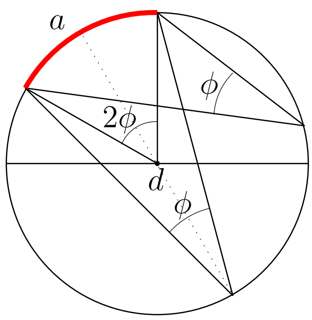

Figure 5: $|a| = \phi d$.

So the arc length of the arc C in Figure 4 equals $\beta - \alpha$, and we have

$$J(C) = \int_{\alpha}^{\beta} (\cos\gamma + \sin\gamma)\, d\gamma \geq \int_{\alpha}^{\beta} 1\, d\gamma = \beta - \alpha\,,$$

the inequality follows from $\cos\gamma + \sin\gamma \geq 1$ for $\gamma \in [0, \frac{\pi}{2}]$. □

The integral form for the jagged length also has a geometric interpretation. Let us consider a right angle with slope γ contained in the halfcircle, as shown in Figure 4. The length of the two sides of the right angle equals $\cos\gamma + \sin\gamma$. If we define

$$C_\gamma := \begin{cases} \text{length of the right angle} & \text{if its apex is contained in } C \\ 0 & \text{otherwise} \end{cases}$$

we obtain the nice form

$$J(C) = \int_0^{\frac{\pi}{2}} C_\gamma \, d\gamma \, .$$

This form is used in the proof of the next lemma.

Lemma 3 *Let D be a line segment, and let P be a surrounding polygon such that P and the angle hull $\mathcal{AH}(D)$ with respect to P touch only in some of P's vertices; see Figure 6. Then the arc length of $\mathcal{AH}(D)$ with respect to P from one endpoint of D to the other is less than $2|D|$.*

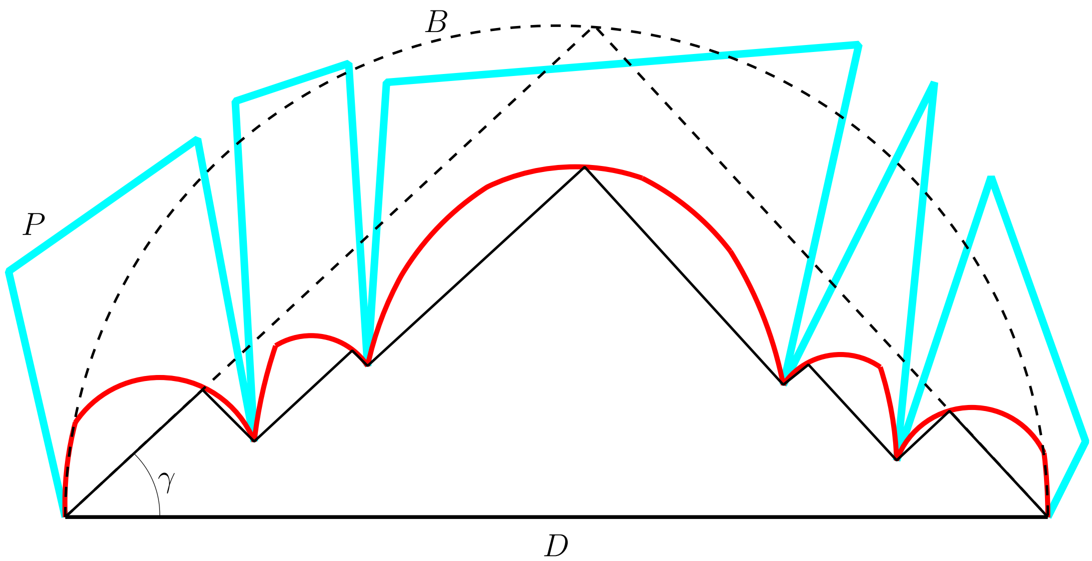

Figure 6: For a line segment D we have $J(\mathcal{AH}(D)) = J(B) = 2|D|$.

Proof. By Lemma 2, the arc length of $\mathcal{AH}(D)$ is certainly shorter than the jagged length of $\mathcal{AH}(D)$, i. e., the sum of the jagged lengths of all circular arcs of $\mathcal{AH}(D)$, and we obtain

$$\begin{aligned}\text{length}(\mathcal{AH}(D)) &\leq J(\mathcal{AH}(D)) = \sum_{C\in\mathcal{AH}(D)} J(C) \\ &= \sum_{C\in\mathcal{AH}(D)} \int_0^{\frac{\pi}{2}} C_\gamma \, d\gamma = \int_0^{\frac{\pi}{2}} \left(\sum_{C\in\mathcal{AH}(D)} C_\gamma \right) d\gamma \, .\end{aligned}$$

But for any angle γ the sum over the lengths of the right angles of slope γ which are contained in the halfcircles of the different circular arcs of $\mathcal{AH}(D)$ is equal to the length, B_γ, of the big right angle in the halfcircle B spanned by the two endpoints of D, which means that

$$\int_0^{\frac{\pi}{2}} \left(\sum_{C\in\mathcal{AH}(D)} C_\gamma \right) d\gamma = \int_0^{\frac{\pi}{2}} B_\gamma \, d\gamma = J(B) = 2|D| \, .$$

□

Note that in the proof of Lemma 3 the halfcircle B does not depend on P and $J(\mathcal{AH}(D)) = J(B)$ therefore means that the jagged lengths of the angle hulls of D for different surrounding polygons P are all identical! We may also say that we have bounded the length of the angle hull with respect to a surrounding polygon P by the jagged length of the angle hull without obstacles.

Lemma 4 *The statement of Lemma 3 remains true if D is a convex chain instead of a line segment.*

Proof. We consider a convex chain, D, and a surrounding polygon, P, such that P and the angle hull $\mathcal{AH}(D)$ with respect to P touch only in some of P's vertices.

We make a construction similar to the proof of Lemma 3. For an angle γ we find the tangent to D with that slope. Starting with the touching vertex we go into direction γ until we hit an arc of the angle hull, then we turn by a right angle and go to the vertex of P (or D) that co-spans the current arc. Here we turn back to the original direction and continue accordingly to obtain a connected chain of right angles, see Figure 7.

This chain has the same length as the two sides of the "unfolded" big right angle of slope γ which generates the angle hull without obstacles. As before, this shows that the jagged length of the angle hull does not depend on P.

Now it is not difficult to see how long it really is. Consider the set of halfcircles spanned by the segments of D. Analogouly to the previous construction, we can construct the chain of right angles of slope γ below these halfcircles which again has the same length. But these right angles represent the jagged lengths of the isolated segments, and each of them equals twice the length of the segment, by

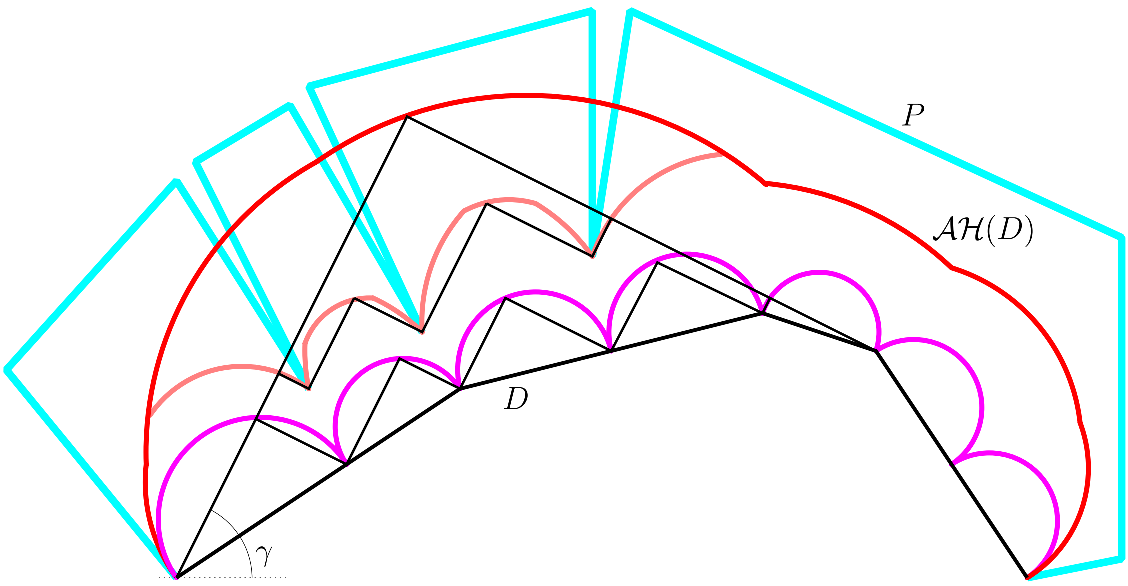

Figure 7: Three chains of right angles which are all of the same length.

Lemma 3. Therefore the total length of the angle hull of D is less than twice the length of D. □

To obtain our main result we need to consider an arbitrary surrounding polygon P that influences the angle hull not only with acute reflex vertices but also with its edges.

Theorem 1 *Let P be a simple polygon containing a relatively convex polygon D. The arc length of the boundary of the angle hull, $\mathcal{AH}(D)$, with respect to P is less than 2 times the length of D's boundary. This bound is tight.*

The bound also holds if there are several obstacles instead of P that influence the angle hull and also if their boundaries consist of arbitrary curves.

Proof. Each convex chain of D can be treated separately because the angle hull must pass through the reflex vertices of D.

First, we consider the angle hull $\mathcal{AH}_1(D)$ with respect to only the vertices of P as obstacle points. Its arc length is less than $2|D|$, by Lemma 4.

Now also the edges come into play. The angle hull $\mathcal{AH}_2(D)$ with respect to the whole of P contains circular arcs and some pieces of P's edges, for an example see Figure 2. The circular arcs of $\mathcal{AH}_2(D)$ are also part of $\mathcal{AH}_1(D)$.

For every piece of an edge which contributes to $\mathcal{AH}_2(D)$, the piece's two endpoints are also on the boundary $\mathcal{AH}_1(D)$. Therefore, $\mathcal{AH}_2(D)$ can only be shorter than $\mathcal{AH}_1(D)$.

The bound is tight by Lemma 1.

The proof easily generalizes to the case of several obstacles around D that influence the angle hull instead of P. In fact, we have never really used the fact that the parts of P that are touching the angle hull are connected by edges of P. And if we have several obstacles, we can always connect them to a single one by edges which do not influence the angle hull.

The proof carries over to arbitrary curves by approximation of these curves with polygons. □

The following variation of Theorem 1 for an "incomplete angle hull" is used for analyzing the exploration strategy in Part I of this paper [3].

Corollary 1 *Consider a convex chain, D, from s to t and its angle hull from s to some point g. The jagged length of this part of the angle hull is bounded by twice the length of the shortest path around D from s to g.*

Proof. Let m be the last segment of D and let α be the angle between m and the last segment of the shortest path from s to g; see Figure 8.

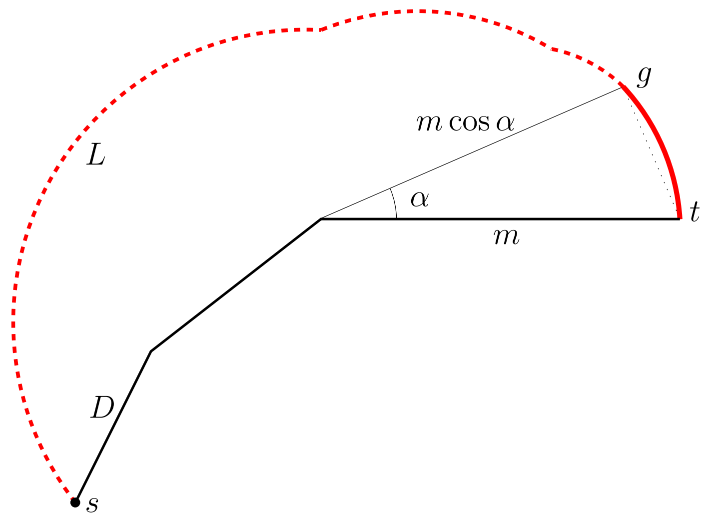

Figure 8: The cut of r_1 is reached at point g.

The jagged length of the angle hull from g to t equals $m(1 + \sin\alpha - \cos\alpha)$. The path, L, from the base point to g can therefore be estimated in the following way, using Theorem 1.

$$\begin{aligned} L &\leq 2|D| - m(1 + \sin\alpha - \cos\alpha) \\ &= 2(|D| - m + m\cos\alpha) + m(1 - \cos\alpha - \sin\alpha) \\ &\leq 2(|D| - m + m\cos\alpha) \end{aligned}$$

The last inequality holds because of $0 \leq \alpha \leq \frac{\pi}{2}$. But $|D| - m + m\cos\alpha$ is exactly the length of the shortest path from the base point to g. □

4 The angle hull in the plane

For the sake of completeness we now consider the outdoor setting, i. e., when the photographer's path and the visibility is not obstructed by a surrounding polygon. Here we get a smaller ratio between the angle hull and the perimeter of D.

Lemma 5 *For a convex polygon, D, the boundary of its angle hull, $\mathcal{AH}(D)$, is at most $\frac{\pi}{2}$ times longer than the boundary of D.*

Proof. As described before, the boundary of $\mathcal{AH}(D)$ is the locus of the apex of a right angle which is rotated around D while the angle's two sides touch D from the outside. It consists of circular arcs, each one spanned by two vertices of D.

Consider a circle with diameter d and a circular arc a on its boundary. Two lines passing through the endpoints of the arc and an arbitrary third point on the boundary always intersect in the same angle α, by the "generalized Thales' theorem". For the length of a, we have $|a| = \alpha d$.

As we have seen in the proof of Lemma 2 and Figure 5, the length of one such circular arc of $\mathcal{AH}(D)$, a, is the distance between the two spanning vertices, d, times the angle, α, by which the right angle rotates to generate a; see Figure 9.

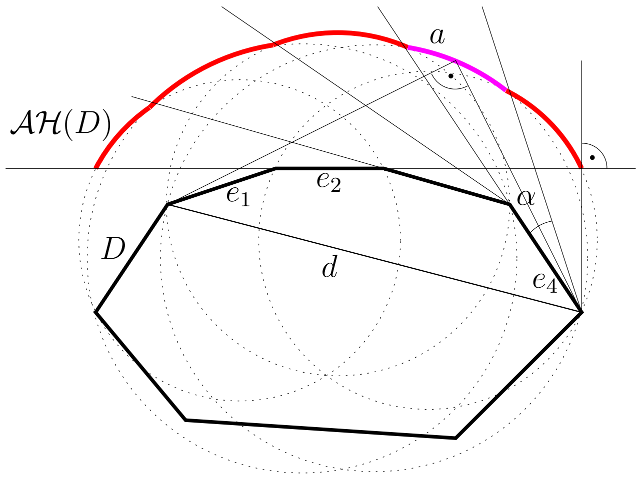

Figure 9: The length of arc a equals $\alpha d \leq \alpha(|e_1| + \cdots + |e_4|)$; edge e_2 contributes only to the arcs above the horizontal line.

The distance, d, is clearly not greater than the length of the boundary of D between the two spanning vertices. The length, L, of the whole boundary of $\mathcal{AH}(D)$ is the sum over all such circular arcs a_i and we get a formula of the following form.

$$L = \sum_i a_i \leq \sum_i \alpha_i (e_{i_s} + e_{i_s+1} + \cdots + e_{i_t})$$

Here, α_i is the corresponding angle for arc a_i and $e_{i_s}, \ldots, e_{i_t}$ denote the edge lengths of D between the two spanning vertices of a_i. Each e_j contributes only to the part of the boundary of $\mathcal{AH}(D)$ that is separated from D by the line through e_j; see edge e_2 in Figure 9. In order to generate this boundary part, the right angle rotates by exactly 90°. Thus, after re-ordering the sum on the right hand side we have

$$L \leq \sum_j e_j(\alpha_{j_s} + \cdots + \alpha_{j_t}),$$

where the angles $\alpha_{j_s}, \ldots, \alpha_{j_t}$ belonging to edge e_j add up to a right angle. Thus, we obtain

$$L \leq \frac{\pi}{2} \sum_j e_j$$

which concludes the proof. □

For D being a line segment, $\mathcal{AH}(D)$ is a circle with diameter D, so the upper bound of $\frac{\pi}{2} = 1.571\ldots$ is tight. Other examples with this property are triangles with only acute angles, or rectangles.

5 Generalizations

Now let's figure out what happens if the photographer changes the lens and uses one with fixed angle ϕ. The generalized angle hull, $\mathcal{AH}_\phi(D)$, of a set D still consists of circular arcs and it is obvious that it will be the bigger the smaller ϕ is. Its perimeter will tend to ∞ if ϕ approaches 0, at least for the unrestricted case, and tends to the perimeter of $\mathcal{RCH}(D)$ if ϕ comes close to 180°.

Without obstacles, we can still make use of the generalized Thales' theorem. We only have to deal with the length of the chord of a circle instead of its diameter. The length of one circular arc of $\mathcal{AH}_\phi(D)$ is the distance between the two spanning vertices, d, times $\alpha/\sin\phi$, where α is the angle by which the fixed angle ϕ rotates to generate the arc. For the perimeter, L, of $\mathcal{AH}_\phi(D)$ we now obtain

$$L \leq \sum_j e_j \frac{\alpha_{j_s} + \cdots + \alpha_{j_t}}{\sin\phi} = \frac{\pi - \phi}{\sin\phi} \sum_j e_j$$

in the same way as before. Remark that the angle ϕ rotates by $\pi - \phi$ to generate the circular arcs contributed by a certain edge e_j.

The factor $(\pi - \phi)/\sin\phi$ is tight, since it is attained in the case of D being a line segment.

The generalization in the indoor setting is not more difficult. For the jagged length of a circular arc C spanned with fixed angle ϕ by a chord of length 1 from angle α to angle β we have

$$J_\phi(C) = (\sin\beta - \sin\alpha - \cos\beta + \cos\alpha)\frac{\cos\phi + 1}{\sin^2\phi}$$

from which a tight factor of $2(\cos\phi + 1)/\sin^2\phi$ follows.

The following interesting mathematical result is a direct consequence of the generalized factor.

Theorem 2 *Let $f : [0,1] \to \mathbf{R}^+$ be a continuous and rectifiable function satisfying $f(0) = 0$ and $f(1) = 0$. For a point p on the graph G of f let ϕ_p denote the maximum angle with apex p whose sides intersect with the X-axis but not with G; see Figure 10. Let $\phi = \inf_{p \in G} \phi_p$. Then the arc length of G is not greater than $2(\cos\phi + 1)/\sin^2\phi$. This bound is tight.*

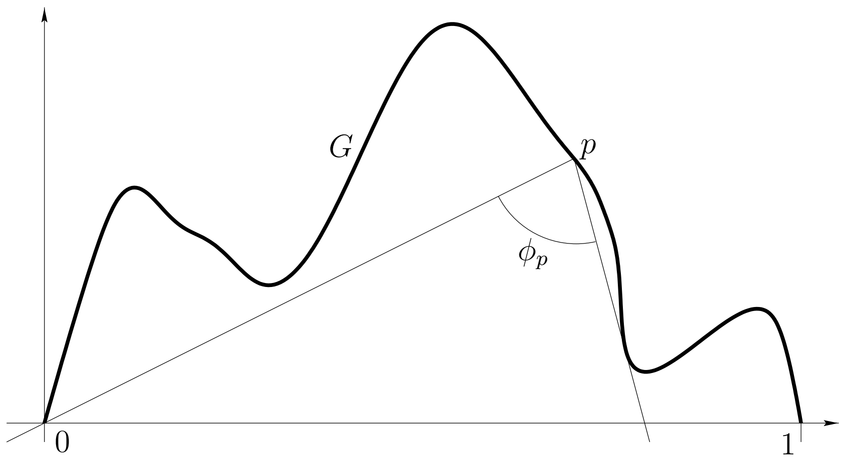

Figure 10: The arc length of a curve can be bounded in terms of its smallest photographer's angle from below.

6 Conclusions

We have introduced a new type of hull operator that is interesting in its own right and suits us well in analyzing the on-line exploration strategy for simple polygons presented in Part I of this paper [3].

Here we have analyzed the perimeter of the angle hull, $\mathcal{AH}(D)$, in terms of the perimeter of the region D. A number of interesting questions remain open: If we consider a subset of D, is the perimeter of its angle hull always shorter than the perimeter of $\mathcal{AH}(D)$? Does the iterated construction of the angle hull approximate a circle? How can angle hulls be generalized, and analyzed, in three dimensions?

References

[1] O. Aichholzer, F. Aurenhammer, C. Icking, R. Klein, E. Langetepe, and G. Rote. Generalized self-approaching curves. In *Proc. 9th Annu. Internat. Sympos. Algorithms Comput.*, volume 1533 of *Lecture Notes Comput. Sci.*, pages 317–326. Springer-Verlag, 1998.

[2] F. Hoffmann, C. Icking, R. Klein, and K. Kriegel. Moving an angle around a region. In *Proc. 6th Scand. Workshop Algorithm Theory*, volume 1432 of *Lecture Notes Comput. Sci.*, pages 71–82. Springer-Verlag, 1998.

[3] F. Hoffmann, C. Icking, R. Klein, and K. Kriegel. The polygon exploration problem I: A competitive strategy. Technical Report 241, Department of Computer Science, FernUniversität Hagen, Germany, 1998.

[4] C. Icking and R. Klein. Searching for the kernel of a polygon: A competitive strategy. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 258–266, 1995.

[5] C. Icking, R. Klein, and E. Langetepe. An optimal competitive strategy for walking in streets. In *Proc. 16th Sympos. Theoret. Aspects Comput. Sci.*, volume 1563 of *Lecture Notes Comput. Sci.*, pages 110–120. Springer-Verlag, 1999.

[6] C. Icking, R. Klein, and L. Ma. How to look around a corner. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 443–448, 1993.

[7] G. Rote. Curves with increasing chords. *Math. Proc. Camb. Phil. Soc.*, 115:1–12, 1994.

[8] R. Seidel. Personal communication, 1997.