# Self-Approaching Curves*

Christian Icking** Rolf Klein** Elmar Langetepe**

**Abstract**

We present a new class of curves which are *self-approaching* in the following sense. For any three consecutive points $a$, $b$, $c$ on the curve the point $b$ is closer to $c$ than $a$ to $c$. This is a generalisation of *curves with increasing chords* which are *self-approaching* in both directions. We show a tight upper bound of $5.3331\ldots$ for the length of a self-approaching curve over the distance between its endpoints.

**Keywords.** Curves with increasing chords, self-approaching curves, convex hull, detour, arc length.

## 1 Introduction

Let $C$ be an oriented curve in the plane. We call $C$ *self-approaching*, iff for any three consecutive points $a$, $b$, $c$ in oriented order on $C$, the inequality

$$d(a,c) \geq d(b,c)$$

holds. In other words, while walking along a self-approaching curve, one gets closer to each point of the curve that has not yet been passed.

The question studied in this paper is the following: Is this property strong enough to ensure that the *detour* a self-approaching curve makes while running from $a$ to $b$, i. e. the ratio of the length of the connecting curve segment over the Euclidian distance from $a$ to $b$, can be bounded, independently of $a$, $b$, and the curve $C$?

This problem has recently arisen in computational geometry, in analysing the performance of an on-line navigation strategy for a mobile robot; see [3].

There is an interesting connection between self-approaching curves and *curves with increasing chords* that are defined by the property

$$d(a,d) \geq d(b,c)$$

for any four consecutive curve points $a$, $b$, $c$, $d$. Namely, a curve has increasing chords iff it is self-approaching in both directions. In [2] the problem analogous to ours has been posed for curves with increasing chords. A solution has been provided by Rote in [4]. He cuts a curve with increasing chords into small pieces, sorts them

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by slope and shows that the reassembled curve has again increasing chords. As it is also convex, by construction, its length can easily be bounded. This way, Rote shows that $2\pi/3 \approx 2.094$ is the sharp upper bound for the detour of planar curves with increasing chords.

Unfortunately, Rote's method cannot be applied to self-approaching curves, because this weaker property is, in general, not preserved if segments are sorted by slope; see Figure 1 for an example.

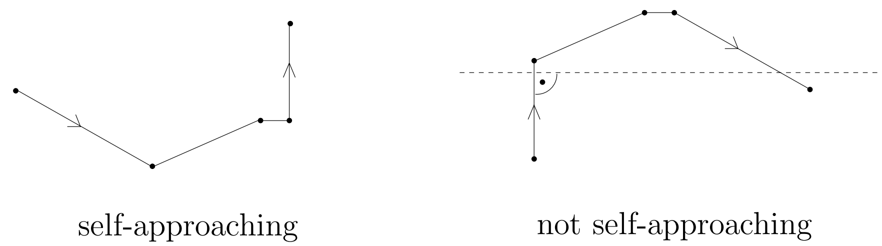


Figure 1: If the segments of a self-approaching polygonal line are sorted by slope, the result need not remain self-approaching.

In fact, establishing a sharp upper bound for the detour of self-approaching curves seems to be more complicated.

In this paper, we proceed as follows. First, in Theorem 4 we prove that the length of a self-approaching curve cannot exceed the perimeter of its convex hull. Next, we circumscribe a self-approaching curve by a simple, closed convex curve whose length can be easily computed. This yields an upper bound for the perimeter of the convex hull; see Theorem 5. Finally in Theorem 6, we demonstrate that the resulting bound of $5.3331\ldots$ is in fact the sharp upper bound for the detour of self-approaching curves.

## 2 Definitions and properties

The curves considered here are assumed to be piecewise smooth curves in the plane. For a curve $C$ and a point $a$ inside a smooth piece of $C$, the *tangent* to $C$ at $a$ and the *normal* to $C$ at $a$, which is perpendicular to the tangent, are uniquely determined. Let $a$ be a point of $C$ such that two smooth pieces of $C$ meet at $a$. The two normals $N_1$ and $N_2$ to the corresponding smooth pieces at $a$ define a set of lines, each line of this set is regarded as a normal to $C$ at $a$; see Figure 2.

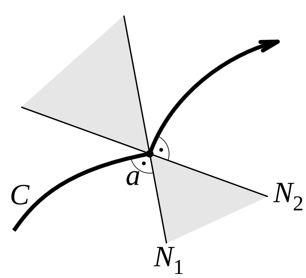


Figure 2: The bundle of lines defined as normals to $C$ at $a$.

Let $d(a, b)$ denote the Euclidean distance between two points $a$ and $b$. For two points $a \le b$ ($a < b$) on a directed curve $C$, $C^{\ge a}$ ($C^{>a}$) denotes the part of $C$ from $a$ to the end (without $a$), $C[a,b]$ means the part of $C$ between $a$ and $b$, and length$(C[a,b])$ means its arc length.

**Definition 1** An oriented curve is called *self-approaching* if the inequality

$$d(a,c) \geq d(b,c)$$

is fulfilled for any three consecutive points $a$, $b$, $c$ on the curve.

Let $C$ be an oriented curve from $a$ to $b$. Then the quantity

$$\frac{\text{length}(C[a,b])}{d(a,b)}$$

is called the *detour* of a curve from $a$ to $b$. The detour of a curve from $a$ to $b$ is the reciprocal of the *minimum growth rate* used in [4].

First, we give two equivalent definitions for self-approaching curves. The following lemma shows that the self-approaching property is equivalent to the fact that for any point $a$ on the curve the rest of the curve lies fully on one side of any normal to $C$ at $a$. We call this the *normal property.*

**Lemma 2** *An oriented curve $C$ is self-approaching iff any normal to $C$ at any point $a$ does not cross $C^{>a}$.*

**Proof.** The *normal property* means that in point $a$ we move closer or maintain the distance to every point in $C^{>a}$. This property holds continuously, so for any three consecutive points the self-approaching property holds.

If the *normal property* is not fulfilled then there exists a point $a$ such that a normal to $C$ at $a$ crosses $C^{>a}$ in $c'$. Then in $a$ we move away from some points in $C^{>c'}$. So there are points $b \in C^{>a}$ and $c \in C^{>c'}$ for which the self-approaching property is not true. □

Now we give another equivalent definition of the self-approaching property which we call the *right angle property*.

**Lemma 3** *An oriented curve $C$ is self-approaching iff for any point $a$ on $C$ there is a right angle at point $a$ which contains $C^{\geq a}$.*

**Proof.** Let $a$ be a point on a self-approaching curve $C$ and consider the two tangents $T_1$ and $T_2$ from $a$ to $C^{\geq a}$ such that $C^{\geq a}$ lies between $T_1$ and $T_2$ which span an angle $\varphi$. If one of $T_1$ and $T_2$ is a tangent to $C$ at $a$ then $\varphi \leq 90°$ follows from the *normal property* (Lemma 2), see Figure 3 (i). Otherwise there are two points $b, c \in C^{>a}$ with $b \in T_1$ and $c \in T_2$, see Figure 3 (ii). Let us assume that $b$ appers before $c$ on $C$. Then from Definition 1 we have $d(a,c) \geq d(b,c)$, in other words $\overline{bc}$ is not the largest edge of the triangle $abc$, which means that $\varphi \leq 90°$.

If a curve $C$ is not self-approaching, then due to Lemma 2 the *normal property* is not fulfilled, i. e. there is a point $a$ such that a normal to $C$ at $a$ crosses $C^{>a}$ in $c'$. Then in $a$ we move away from some points in $C^{>c'}$. So there are points $b \in C^{>a}$ and $c \in C^{>c'}$ for which the *right angle property* is not true. □

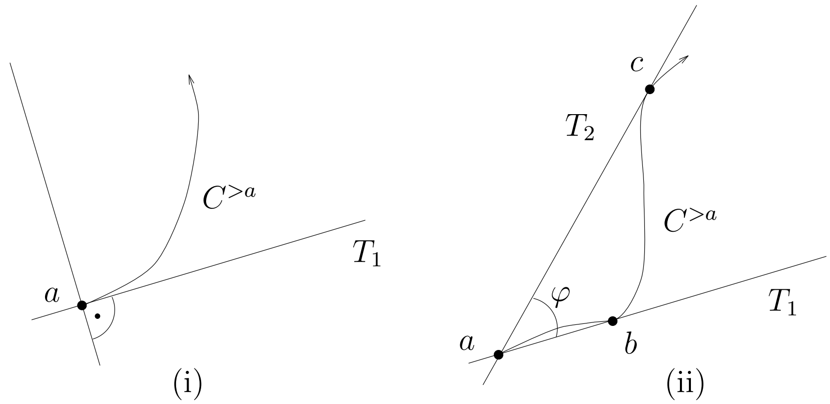


Figure 3: For any point $a$ on $C$, the rest of the curve is included in a right angle at $a$.

**Example.** The logarithmic spiral, directed to the center, is an interesting example for a self-approaching curve. In polar coordinates it is the set of all points $(\varphi, e^{\varphi \cot \alpha})$ with constant angle $\alpha < 90°$ between the tangent and the radius to each point on the curve, see Figure 4. It is self-approaching if $\alpha$ fulfills

$$\alpha \leq \arctan\left(\frac{3\pi}{2\,\mathrm{W}(\frac{3}{2}\pi)}\right) \approx 74.66°$$

in which $W$ denotes Lambert's W function [1] defined by the functional equation $W(x)\,e^{W(x)} = x$. Figure 4 shows the limiting case where the normal at any point is tangent to the rest of the curve.

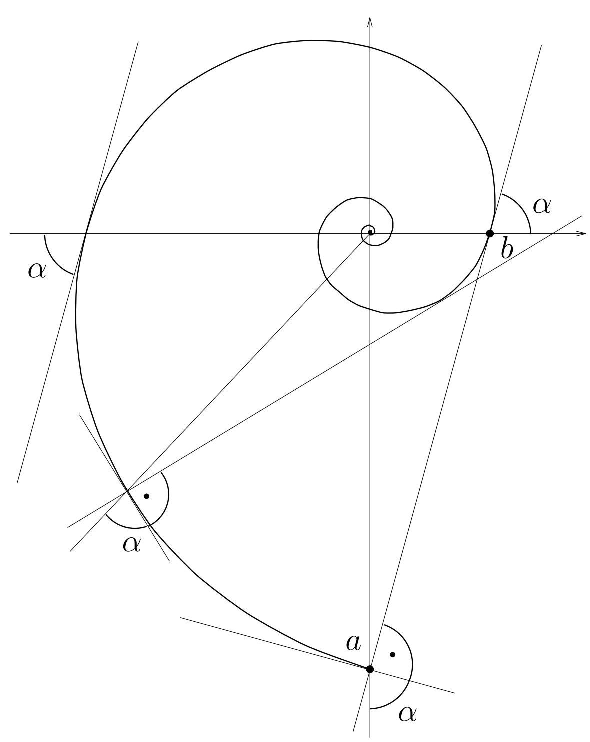


Figure 4: The narrowest self-approaching logarithmic spiral.

This special curve is in a sense the narrowest self-approaching logarithmic spiral. One can show that its detour equals $1/\cos\alpha_{\max} \approx 3.78$, but despite its optimized form there are other self-approaching curves with a bigger detour. For example, the simple curve shown in Figure 5 has a detour of $\pi + 1$.

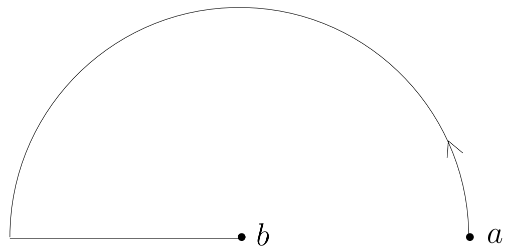


Figure 5: A simple self-approaching curve with a detour of $\pi + 1$.

However, there is something interesting about the logarithmic spiral. Let us suppose that we fix a string at point $b$ of Figure 4 and attach a pencil at point $a$. Now we move the pencil clockwise holding the string taut. Then the pencil draws the spiral while the string wraps around the inner part of the spiral. (Therefore this curve is its own involute.) This implies that the string is of the same length as the inner part of the spiral. Consequently, the total length of the spiral equals the perimeter of its convex hull! This fact will now be generalized to arbitrary self-approaching curves.

# 3 Analysing the detour of self-approaching curves

First we show that the length of self-approaching curves are bounded by the perimeter of their convex hull. Then we estimate the perimeter of their convex hull and prove, by giving an example, that the bound is tight.

Let $\text{ch}(C)$ denote the convex hull of a curve $C$ and $\text{per}(C)$ the length of the perimeter of $\text{ch}(C)$. For two points $a$ and $b$ let $R(a, b)$ denote the ray starting at $a$ and passing through $b$.

**Theorem 4** *The length of a self-approaching curve $C$ is less than or equal to the perimeter,* $\text{per}(C)$*, of its convex hull.*

**Proof.** The length of a curve $C$ is, by definition, the supremum of the lengths of all polygonal chains with vertices on $C$ in the same order as they appear on $C$. Therefore, an upper bound for the length of all such chains is also an upper bound for the length of $C$.

We take an arbitrary polygonal chain $Q$ whose vertices lie on $C$ in the same order. By induction on the number of vertices of $Q$, we will prove that $Q$ is shorter than the perimeter, $\text{per}(Q)$, of its convex hull, $\text{ch}(Q)$, which in turn is bounded by $\text{per}(C)$. Note that the vertices of $\text{ch}(Q)$ are also vertices of $Q$ and are therefore points on $C$.

The assertion is true for $Q$ being a line segment, so let us assume that $Q$ has at least three vertices, the first two are called $a$ and $b$. The induction hypothesis is that $\text{length}(Q^{\geq b}) \leq \text{per}(Q^{\geq b})$.

We distinguish two cases depending on whether $b$ lies on the boundary of $\text{ch}(Q)$ or not.

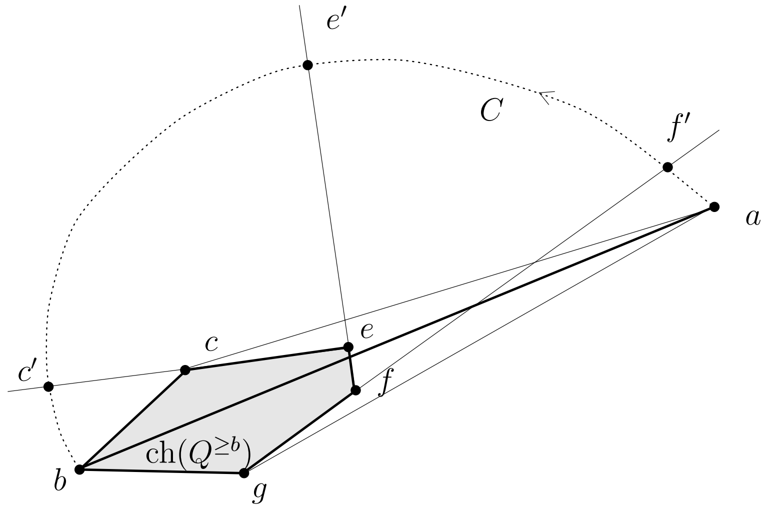


Figure 6: $d(c, a) + d(g, a) \geq d(g, f) + d(f, e) + d(e, c) + d(a, b)$.

**Case 1.** The point $b$ is on the boundary of ch($Q$). We have a situation as depicted in Figure 6. When passing from ch($Q^{\geq b}$) to ch($Q$), the convex hull changes as follows. The segments $\overline{gf}$, $\overline{fe}$ and $\overline{ec}$ which belong to ch($Q^{\geq b}$) are replaced by the segments $\overline{ca}$ and $\overline{ga}$. Since length($Q^{\geq b}$) + $d(a, b)$ = length($Q$) it suffices to prove that

$$d(c, a) + d(g, a) \geq d(g, f) + d(f, e) + d(e, c) + d(a, b).$$

We do not know which way $C$ takes from $a$ to $b$ but there are either points $f' \in R(g, f)$, $e' \in R(f, e)$ and $c' \in R(e, c)$ in exactly this order on $C[a, b]$ or there are points $c' \in R(c, e)$, $e' \in R(e, f)$ and $f' \in R(f, g)$ in this order on $C[a, b]$. W.l.o.g. we assume the first case.

While the curve $C$ moves from $a$ to $f'$ it gets closer to $f$ because $f'$ arises before $f$ on $C$. Therefore

$$d(g, a) \geq d(g, f') = d(g, f) + d(f, f')$$

By the same argument $C$ gets closer to $f$ while it runs from $f'$ to $e'$. Therefore

$$d(f, f') \geq d(f, e') = d(f, e) + d(e, e')$$

Similarly we have $d(e, e') \geq d(e, c) + d(c, c')$ and also $d(c, c') \geq d(c, b)$. Altogether we conclude

$$d(c, a) + d(g, a) \geq d(g, f) + d(f, e) + d(e, c) + d(c, b) + d(c, a).$$

The triangle inequality $d(c, b) + d(c, a) \geq d(a, b)$ finishes the proof. This also works for $c = b$ or $g = b$.

Notice that this generalizes to any number of vertices of ch($Q^{\geq b}$), instead of $e$, $f$ and $c$.

**Case 2.** The point $b$ is not on the boundary of ch($Q$). Then $a$ must lie in the wedge included between the prolongations of the adjacent edges of ch($Q^{\geq b}$) at $b$, see Figure 7. The neighbouring vertices of $b$ in ch($Q^{\geq b}$) are called $c$ and $e$. The angle $\varphi$ between $\overline{bc}$ and $\overline{be}$ is less than or equal to 90° because of the right angle property (Lemma 3).

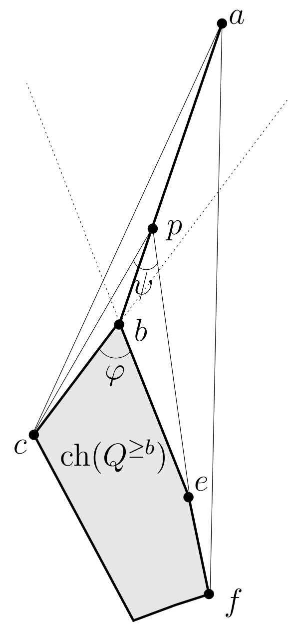


Figure 7: $d(c,p) + d(e,p) \geq d(b,p) + d(b,c) + d(b,e)$

We assume a situation as in Figure 7 where the points $b$ and $e$ of $\mathrm{ch}(Q^{\geq b})$ are not on the boundary of $\mathrm{ch}(Q)$. Let $p$ be the intersection point of $\overline{a\,b}$ and $R(e,f)$. Using the induction hypothesis and the fact $\varphi \leq 90°$ we first show that $\mathrm{length}(Q^{\geq p})$ is not greater than $\mathrm{per}(Q^{\geq p})$. Passing from $\mathrm{ch}(Q^{\geq b})$ to $\mathrm{ch}(Q^{\geq p})$, the convex hull changes as follows. The segments $\overline{b\,c}$ and $\overline{b\,e}$ which belong to $\mathrm{ch}(Q^{\geq b})$ are replaced by the segments $\overline{p\,c}$ and $\overline{p\,e}$ of $\mathrm{ch}(Q^{\geq p})$. Since $\mathrm{length}(Q^{\geq p})$ is equal to $\mathrm{length}(Q^{\geq b}) + d(b,p)$ it is sufficient to show that

$$d(c,p) + d(e,p) \geq d(b,p) + d(b,c) + d(b,e)$$

holds, this is exactly the conclusion of Lemma 7 in the appendix on page 12; the proof is elementary but lengthy.

Since we can use the assumption that $\mathrm{length}(Q^{\geq p}) \leq \mathrm{per}(Q^{\geq p})$ and the fact $\psi \leq \varphi \leq 90°$ the same argument holds also for $Q^{\geq a} = Q$ and also for the case that more vertices of $\mathrm{ch}(Q^{\geq b})$ than only $b$ and $e$ do not reappear as vertices in $\mathrm{ch}(Q)$. This concludes the proof. □

In the following, for two points $p$ and $q$ let $\mathrm{circ}_p(q)$ denote the circle with center $p$ passing through $q$.

As an immediate consequence of Theorem 4, we have an upper bound of $2\pi$ for the detour of self-approaching curves, because any such curve from point $a$ to point $b$ must be contained in $\mathrm{circ}_b(a)$. The following theorem refines this argument to a smaller bound, which will be shown to be tight afterwards in Theorem 6.

**Theorem 5** *The perimeter of the convex hull of a self-approaching curve is not greater than*

$$c_{\max} := \max_{\beta \in [0..\frac{\pi}{2}]} \frac{2\beta + \pi + 2}{\sqrt{5 - 4\cos\beta}} \approx 5.3331\ldots$$

*times the distance of its endpoints.*

**Proof.** Let $a$, $f$ denote the first resp. final point of a self-approaching curve $C$. The proof works as follows: We show that $\text{per}(C) \leq c_{\max} d(a,f)$ holds if the curve does *not* cross the line segment $\overline{af}$. If it does, we apply this bound for each subcurve between two successive curve points on $\overline{af}$ and add up the length. Due to the self-approaching property, the curve points on $\overline{af}$ appear in the same order as on $C$. Then we know that the sum of the perimeters of all such subcurves is less than $c_{\max}$ times the distance between $a$ and $f$. Now consider two subsequent subcurves $C_1$ and $C_2$. The two convex sets $\text{per}(C_1)$ and $\text{per}(C_2)$ intersects at least in one point and therefore we conclude $\text{per}(C_1 \cup C_2) \leq \text{per}(C_1) + \text{per}(C_2)$. This argument can be applied successively, so the perimeter of the whole curve is smaller than the sum of the perimeters of all the subcurves which in turn is smaller than $c_{\max}$ times the distance between $a$ and $f$.

So for the rest of this section we may assume that $C$ does not cross the line segment $\overline{af}$. Because of the right angle property(Lemma 3), the whole of $C$ lies between two orthogonal halflines $X$ and $Y$ starting at $a$. W.l.o.g. we assume that the initial part of $C$ lies on the left side of the edge directed from $a$ to $f$, and, if necessary, we rotate $X$ and $Y$ such that the halfline on the other (right) side of $\overline{af}$ touches $C$ at a point $e$, see Figure 8. Let $h$ and $w$ be the height resp. width of the bounding box of $a$ and $f$ according to rectangle with sides parallel to $X$ and $Y$ and with diagonal $\overline{af}$.

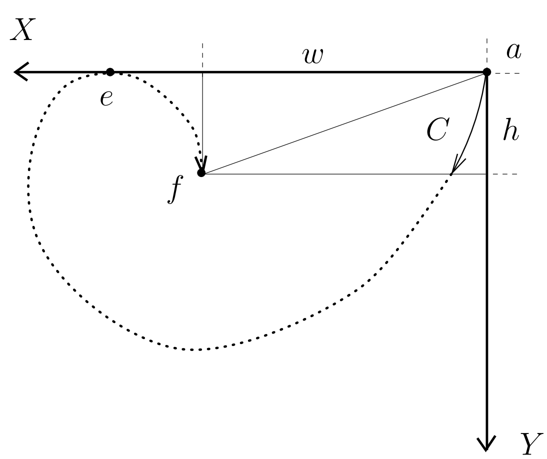


Figure 8: Any self-approaching curve is contained in a wedge of 90°.

We will construct a convex area, $A$, that contains $C$ and we will show that the perimeter of this area divided by $d(a,f) = \sqrt{h^2 + w^2}$ is bounded by $c_{\max}$.

This construction goes as follows, refer to Figure 9. First we know that $C$ is contained in the right angle at $a$, and $C[a,e]$ is contained in $\text{circ}_e(a)$ and $C[e,f]$ is contained in $\text{circ}_f(e)$, due to the self-approaching property.

We also know that $C[a,e]$ must avoid $\text{circ}_f(e)$, but pass around it to reach $e$ because it must not cross $\overline{af}$. We conclude that $\text{circ}_f(e)$ is contained in $\text{circ}_e(a)$.

Now, we will enlarge these circles to a certain extent. Instead of $e$, we use a point $e'$ on $X$ with $d(a,e') \geq d(a,e)$, such that $\text{circ}_{e'}(a)$ still contains $\text{circ}_f(e')$ and touches it in one point $c'$. This is possible because for every position of $e'$ on $X$ with $d(a,e') \geq d(a,e)$, the whole circle $\text{circ}_{e'}(a)$ is always on one side of $Y$, while $\text{circ}_f(e')$ must eventually cross $Y$. Note that $d(e',a) = d(e',c') = d(c',f) + d(f,e') = 2d(f,e')$ holds, in other words the radius of $\text{circ}_{e'}(a)$ equals the diameter of $\text{circ}_f(e')$.

Now let $c$ be the point at which curve $C$ crosses $\overline{f\,c'}$ first. We know that $C^{\leq c}$ is contained in $\text{circ}_e(a) \subseteq \text{circ}_{e'}(a)$, $C^{\geq c}$ is contained in $\text{circ}_f(c) \subseteq \text{circ}_f(c') = \text{circ}_f(e')$. So the curve $C$ is included in the convex area $A$ limited by $\overline{e'\,a}$, the circular arc from $a$ to $c'$ about $e'$, and the halfcircle from $c'$ to $e'$ about $f$, see Figure 9.

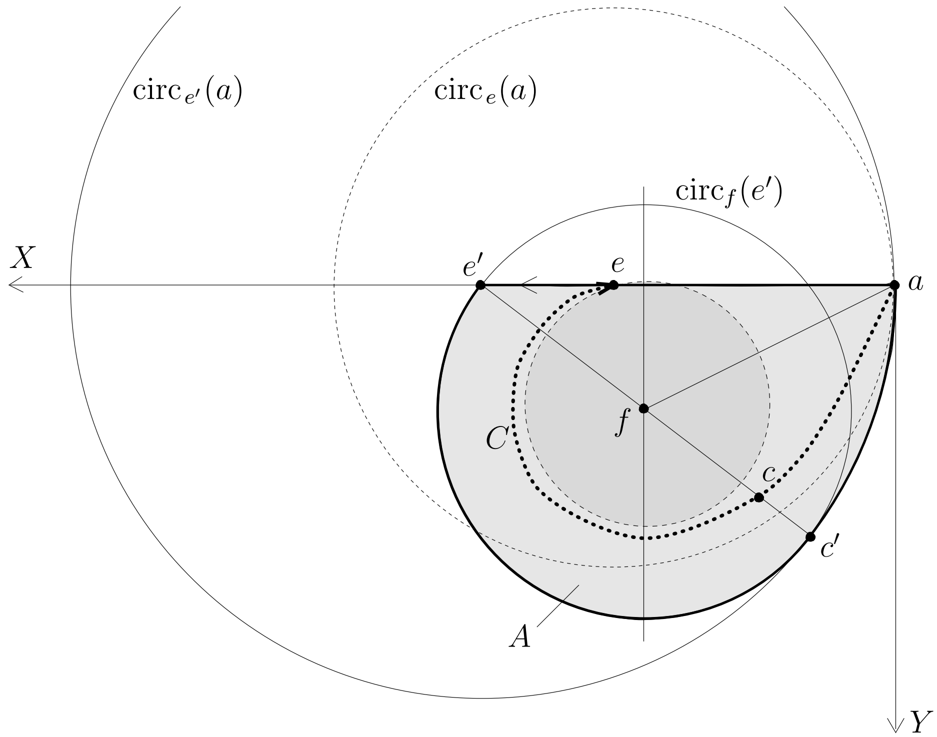


Figure 9: Curve $C$ is contained in the shaded area $A$.

We choose a scale such that $d(f, c') = 1$. Let $\beta$ be the angle between $\overline{e'\,a}$ and $\overline{e'\,c'}$, see Figure 10. Then $d(e', c') = 2 = d(e', a)$, $w = 2 - \cos\beta$ and $h = \sin\beta$. The length of the circular arc from $a$ to $c'$ equals $2\beta$ while the halfcircle from $c'$ to $e'$ has length $\pi$.

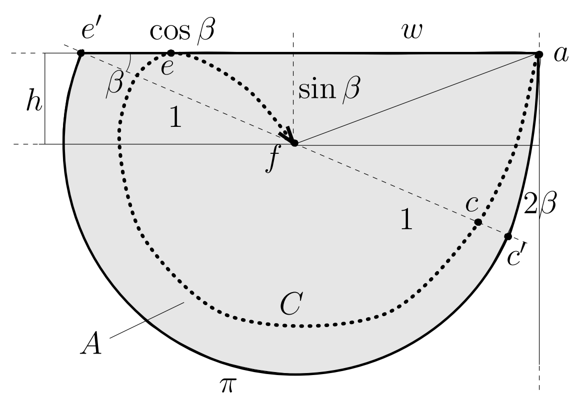


Figure 10: Measuring the perimeter of the convex area $A$.

Then the perimeter of the constructed convex area $A$ divided by $d(a, f) = \sqrt{h^2 + w^2}$ is given by the following function $g$ in $\beta$.

$$g(\beta) := \frac{2\beta + \pi + 2}{\sqrt{\sin^2\beta + (2 - \cos\beta)^2}} = \frac{2\beta + \pi + 2}{\sqrt{5 - 4\cos\beta}}$$

The maximum value, $c_{max}$, of $g(\beta)$ equals aproximately $5.3331\ldots$ for $0 \leq \beta \leq \pi/2$ and is attained at $\beta_{opt} \approx 11.22^\circ$ with $(2\beta_{opt} + \pi + 2)\sin\beta_{opt} = 5 - 4\cos\beta_{opt}$. $\square$

Surprisingly, it turns out that there cannot be a smaller upper bound for self-approaching curves.

**Theorem 6** *The constant $c_{max}$ is a tight upper bound for the detour of self-approaching curves.*

**Proof.** That $c_{max}$ is an upper bound for the detour of self-approaching curves follows directly from Theorem 4 and Theorem 5. To prove tightness we construct a curve with a convex hull similar to the bounding area $A$ in Theorem 5.

As a first step we consider the curve in Figure 11. From the start point, $a$, to the end, $f$, it consists of a circular arc of radius 2 and angle $\beta$, a half circle of radius 1, and a line segment of length 1. This curve is self-approaching, its length equals $2\beta + \pi + 1$ while $d(a, f) = \sqrt{5 - 4\cos\beta}$. The ratio takes on a maximum value of approximately 4.38.

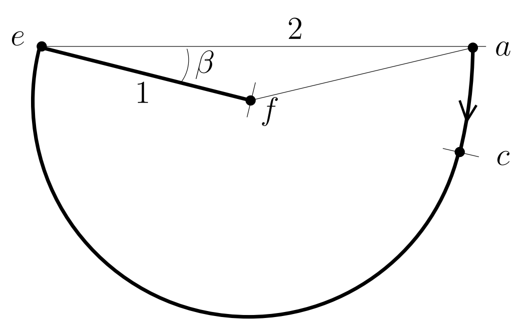


Figure 11: This self-approaching curve has a detour of 4.38.

But there is some room for improvements in the last step, i. e. the line segment from $e$ to $f$. Instead of walking straight from $e$ to $f$ we use a sequence of pieces of small cycloids. (A cycloid is known to be the orbit of a point on the boundary of a rolling circle and it has another cycloid as its involute.) For an odd number $n \in \mathbb{N}$ we can fill a rectangle of height $h$ and width $w = 2nh/\pi$ with $n$ successive congruent pieces of cycloids such that they form a curve from the lower left to the upper right corner, see Figure 12. Each piece is a cycloid generated by a circle of radius $h/\pi$ rolling on a vertical line, and each one is the involute of its successor. The resulting curve is self-approaching, and its length is exactly $2w$ since the length of a piece is twice its width.

Now let us replace the line segment in our first step by such a construction in a rectangle of height $h = \frac{1}{(2n/\pi)+1}$ and width $w = 1 - h = 2nh/\pi$, see Figure 13.

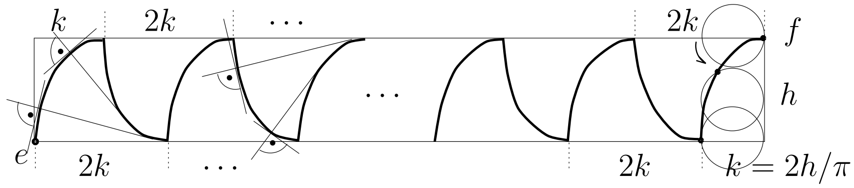


Figure 12: How to fill a rectangle of width $w$ and height $h$ with a self-approaching curve of length $2w$ using pieces of cycloids.

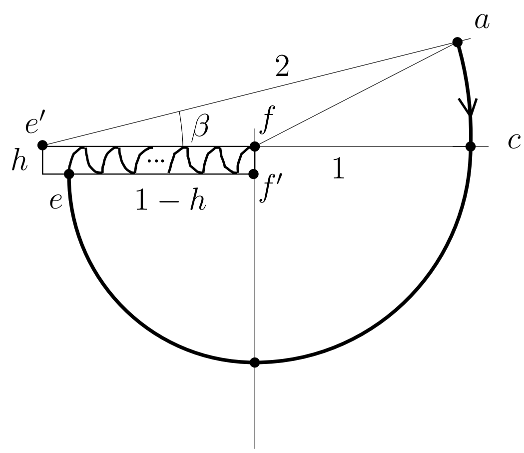


Figure 13: Replacing the line segment by a rectangle that gives room for improvement.

The curve consists of

- a circular arc of radius 2 about $e'$ of length $2\beta$
- a quartercircle of radius 1 about $f$ of length $\pi/2$
- a quartercircle of radius $(1-h)$ about $f'$ of length $(1-h)\pi/2$ and
- a sequence of cycloids from $e$ to $f$ of overall length $2(1-h)$.

Now we choose the constant $\beta = \beta_{opt}$ from Theorem 5 and for $n = 1, 3, 5 \ldots$ we have a sequence of self-approaching curves with a detour of at least

$$\frac{2\beta_{opt} + \pi/2 + (1-h)(\pi/2 + 2)}{\sqrt{5 - 4\cos\beta_{opt}}}$$

which converges to $c_{max}$ for $n \to \infty$ (i. e. $h \to 0$). □

Actually, as $n$ tends to infinity, one could think of the curve of Figure 12 as a thick line segment of length 2, while its endpoints are only distance 1 apart. One might wonder if a factor bigger than 2 can be achieved by a different technique. Note that this is not possible as a direct consequence of Theorem 5. Also note that there are many other ways of constructing the self-approaching thick line segment of length 2.

# A Appendix

**Lemma 7** *Let $v$ be a point inside a triangle $abc$. We connect each vertex to $v$ using segments $l_1 = \overline{bv}$, $r_1 = \overline{cv}$, and $z = \overline{av}$. Let $l_2 = \overline{ab}$ and $r_2 = \overline{ac}$ be two edges of the triangle; see Figure 14. If the angle $\varphi \leq \pi$ between $l_1$ and $r_1$ is less than or equal to $\pi/2$ then for the lengths of the segments $l_1 + r_1 + z \leq l_2 + r_2$ holds.*

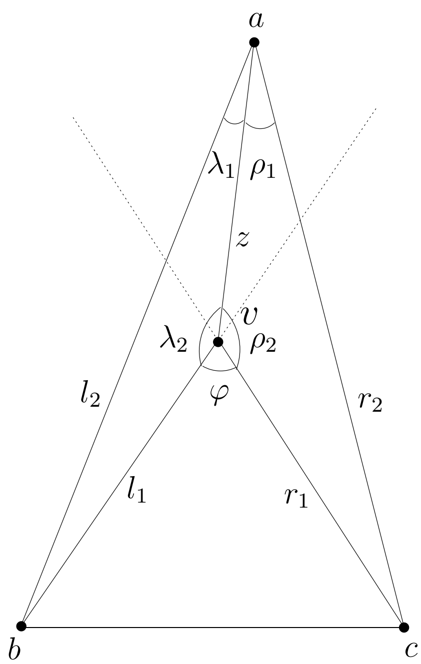


Figure 14: $l_1 + r_1 + z \leq l_2 + r_2$ holds for $\varphi \leq \pi/2$.

**Proof.** The assumption is obviously true for $z = 0$. Let $z \neq 0$. We have to prove the inequality $l_2 - l_1 + r_2 - r_1 \geq z$. Let $\lambda_1 \leq \pi$ be the angle between $l_2$ and $z$, $\lambda_2 \leq \pi$ be the angle between $l_1$ and $z$, $\rho_1 \leq \pi$ be the angle between $r_2$ and $z$ and $\rho_2 \leq \pi$ be the angle between $r_1$ and $z$. Using the law of sines we substitute $l_1$ with $z\frac{\sin\lambda_1}{\sin(\lambda_1+\lambda_2)}$ and $l_2$ with $z\frac{\sin\lambda_2}{\sin(\lambda_1+\lambda_2)}$. We transform $r_1$ and $r_2$ analogously and divide the whole expression by $z$. So we have to show that

$$\frac{\sin\lambda_2 - \sin\lambda_1}{\sin(\lambda_1 + \lambda_2)} + \frac{\sin\rho_2 - \sin\rho_1}{\sin(\rho_1 + \rho_2)} \geq 1$$

is true. We consider some simple transformations:

$$\begin{aligned}
\frac{\sin\lambda_2 - \sin\lambda_1}{\sin(\lambda_1 + \lambda_2)} &= \frac{\sin\lambda_2 - \sin((\lambda_1 + \lambda_2) - \lambda_2)}{\sin(\lambda_1 + \lambda_2)} \\
&= \frac{\sin\lambda_2 - \sin(\lambda_1 + \lambda_2)\cos\lambda_2 + \cos(\lambda_1 + \lambda_2)\sin\lambda_2}{\sin(\lambda_1 + \lambda_2)} \\
&= -\cos\lambda_2 + \underbrace{\frac{(1 + \cos(\lambda_1 + \lambda_2))\sin\lambda_2}{\sin(\lambda_1 + \lambda_2)}}_{(*)} \geq -\cos\lambda_2
\end{aligned}$$

Note that $(*) \geq 0$ holds because of $0 \leq \lambda_1 + \lambda_2 \leq \pi$. Similarly we have $\frac{\sin\rho_2 - \sin\rho_1}{\sin(\rho_1+\rho_2)} \geq -\cos\rho_2$ so it is sufficient to show that $-\cos\lambda_2 - \cos\rho_2 \geq 1$ is true. We conclude $\frac{\lambda_2+\rho_2}{2} = \pi - \frac{\varphi}{2}$ from $\lambda_2 + \rho_2 + \varphi = 2\pi$. Since $\lambda_2$ and $\rho_2$ are inner angles of a triangle we know $\lambda_2, \rho_2 \leq \pi$ and from $\varphi \leq \pi/2$ we conclude $\lambda_2, \rho_2 \geq \pi/2$. Therefore $|\lambda_2 - \rho_2| \leq \pi/2$ is true. Now

$$-\cos\lambda_2 - \cos\rho_2 = -2\cos\left(\frac{\lambda_2+\rho_2}{2}\right)\cos\left(\frac{|\lambda_2-\rho_2|}{2}\right) \geq 1$$

holds since $-\cos\left(\frac{\lambda_2+\rho_2}{2}\right) = \cos\left(\frac{\varphi}{2}\right) \geq \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos\left(\frac{|\lambda_2-\rho_2|}{2}\right) \geq \frac{1}{\sqrt{2}}$ are fulfilled. $\square$

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