

How to Find a Point on a Line within a Fixed Distance

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Abstract

It is well known that on a line, a target point in unknown position can be found by walking a path at most 9 times as long as the distance from the start to the target point, in the worst case. This competitive factor of 9 is optimal. We investigate the case where the target is known to be within a fixed distance, r , of the start point, and determine the optimum competitive factor, $C(r) < 9$, that can be achieved by a competitive strategy $S(r)$, under this additional assumption.

Key words: On-line algorithm, competitive analysis, doubling, navigation, autonomous vehicle, robotics, computational geometry.

1 Introduction

Suppose an autonomous robot is facing an infinite wall. Its task is to get to the other side of the wall. The robot knows that there must be a door somewhere, but it does not know if the door is located to the left or to the right of its current position. For the sake of simplicity we assume that the door is at least one unit away from the robot.²

This problem belongs to the basic tasks in on-line navigation. Baeza-Yates et al. [1] introduced the problem to computational geometry. Among other problems, they considered the case where the robot does not know how far

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² This assumption can be removed by introducing a constant additive term in the definition of competitiveness below, as one usually does in the definition of O .

the door might be away from its current position. They provided the following strategy. First, the robot moves one unit to the right and returns to its start point. Then, it moves 2 units to the left, and returns. Next, the robot walks 4 units to the right, and so on. Each time, the depth of exploration is doubled.

Let us assume that the exploration of depth 2^k misses the door by an ε . Then the robot performs unsuccessful searches of depth $1, 2, \dots, 2^k, 2^{k+1}$ each. The total path length equals twice the sum of these terms, plus another $2^k + \varepsilon$, accounting for the last walk from the start point to the door. Altogether, the robot's path is at most 9 times as long as the distance from its start point to the door.

We call a strategy S for solving all problems P of a class Π *competitive with ratio C* if

$$\text{cost}_S(P) \leq C \cdot \text{cost}_{\text{opt}}(P)$$

holds, for each instance P of Π . Here, the cost $\text{cost}_S(P)$ of solving P by means of strategy S is compared against the cost of an optimal solution. By this definition, the above strategy for finding a point an unknown distance away is competitive with ratio 9.

Interestingly, Baeza-Yates et al. [1] were also able to prove that the ratio of 9 is optimal, i. e. that there is no competitive strategy with a ratio less than 9. This will follow as a special case from our results.

This “doubling” technique has become one of the few paradigms that have so far evolved in the area of on-line navigation. The seminal paper [1] already contains a generalization of this approach to $m > 2$ halflines that intersect at the start point. Icking and Klein [3] have shown how to find a target point in an unknown simple polygon by doubling on the shortest path tree, with an optimal competitive ratio proportional to the number of vertices. A similar approach has recently enabled López-Ortiz and Schuierer [7] to find a target point in a star-shaped polygon with a constant competitive ratio. Blum et al. [2] and Kalyanasundaram and Pruhs [4] are both drawing on the basic idea behind the doubling technique: After an unsuccessful attempt, one can afford doubling the effort as long as it does not exceed a constant times the cost of the optimal solution.³

In their paper [1] Baeza-Yates et al. have also studied the average case for a point drawn from a random distribution over a given interval. It turns out that for certain distributions the optimal strategy may cause the robot to turn

³ Doubling the stake after each loss is a gambling strategy very similar to this approach.

infinitely often. Randomized strategies have been studied by Kao et al. [5].

In this paper we consider the original problem of finding a door in a wall under the following additional assumption: The robot knows that the door is at least one and at most r unit steps away. We are interested in the optimum competitive ratio $C(r)$ that can be achieved under this additional constraint, and in an optimal deterministic strategy $S(r)$ with ratio $C(r)$.

If the *exact* distance d of the door is known one can trivially achieve a competitive ratio of 3 in the following way. The robot first walks d units to the right. If the door is not there the robot returns to the start point and moves d units to the left.

This argument indicates that, for $r \in [1, \infty]$ we should expect $C(r)$ to take values between 3 and 9.

2 Competitive Ratio Versus Reach

In this section we shall investigate the relation between r and $C(r)$ in more detail. Let S be a deterministic competitive strategy for finding a target point on a line provided it is at least one and at most r units away. We call r the *reach* of strategy S . Suppose that S achieves the competitive ratio C .

W.l.o.g. we assume that the robot starts from the origin, and is caused by S to walk f_1 units to the right and to return, then f_2 units to the left, and so on. Thus, S can be described by a sequence (f_1, f_2, \dots) of real numbers $f_i \geq 1$ that eventually grow as large as r .

2.1 A Useful Recurrence

In this subsection we derive some useful facts that will help us establish the relation between the reach of a strategy and its competitive ratio.

For S to have competitive ratio C it is necessary that

$$2f_1 + 1 \leq C \tag{0}$$

holds because the target might be one unit to the left of the start point. Similarly,

$$2 \sum_{i=1}^{n+1} f_i + f_n + \varepsilon \leq C \cdot (f_n + \varepsilon)$$

must be fulfilled for each $n \geq 1$ and each $\varepsilon > 0$ because the exploration of depth f_n may fall short by ε of finding the target. Note that these inequalities are also sufficient for S to be competitive with ratio C .

For $\varepsilon \rightarrow 0$ the latter inequality becomes

$$f_{n+1} \leq Hf_n - \sum_{i=1}^{n-1} f_i \quad (1)$$

with $H := \frac{C-3}{2}$. Analogously, we have

$$f_n \leq Hf_{n-1} - \sum_{i=1}^{n-2} f_i.$$

By plugging this estimate into (1) one obtains a new upper bound for f_{n+1} . This substitution process can be iterated; it yields

$$f_{n+1} \leq a_m f_{n-m} - b_m \sum_{i=1}^{n-1-m} f_i \quad (2)$$

for all $n \geq 1$ and all $0 \leq m \leq n-1$, where $(a_i)_i$ and $(b_i)_i$ are recursively defined by

$$\begin{aligned} a_0 &:= H, & a_{i+1} &:= a_i H - b_i \\ b_0 &:= 1, & b_{i+1} &:= a_i + b_i. \end{aligned}$$

If we let $m = n-1$ in (2) we obtain

$$f_{n+1} \leq a_{n-1} f_1 \quad (3)$$

for each $n \geq 1$. The above recursion can be solved by

$$\begin{aligned} a_i &= vz^i + \bar{v}\bar{z}^i \\ b_i &= v(\bar{z}-1)z^i + \bar{v}(z-1)\bar{z}^i \end{aligned} \quad (4)$$

where

$$v = \frac{Hz - H - 1}{z - \bar{z}}, \quad \bar{v} = \frac{H\bar{z} - H - 1}{\bar{z} - z};$$

here

$$z = \frac{1}{2} \left(H + 1 + \sqrt{(H + 1)(H - 3)} \right)$$

is one solution of the quadratic equation

$$t^2 - (H + 1)t + H + 1,$$

and \bar{z} denotes the other solution.

That these terms do in fact solve the above recursion can be easily verified by induction, using the identities

$$z\bar{z} = H + 1 = z + \bar{z} \quad \text{and} \quad z - \bar{z} = \sqrt{(H + 1)(H - 3)}.$$

How to obtain these solutions can be found in Klein [6].

Now let us assume that $C < 9$, hence $H < 3$. Then z cannot be a real number, so \bar{z} is the complex conjugate of z .

The coefficients a_n can also be expressed by

$$a_n = vz^n + \bar{v}\bar{z}^n = 2\text{Re}(vz^n), \tag{5}$$

where $\text{Re}(w)$ denotes the real part, c , of a complex number $w = c + di$. If we represent complex numbers by points in the plane, multiplication of two numbers entails adding up the corresponding angles they form with the positive X -axis. Since in (5) z is not real, its angle is not equal to 0. Consequently, there exists a smallest natural number s such that vz^s lies in the left halfplane $\{X < 0\}$, so that a_s becomes negative. We infer from (3)

$$f_{s+2} \leq a_s f_1 < 0. \tag{6}$$

For a strategy S of infinite reach the numbers f_i must tend to ∞ and not ever become negative. Therefore, we cannot have $C < 9$. So we have obtained the result by Baeza-Yates et al. [1].

Theorem 1 *For a strategy of infinite reach, 9 is the best competitive ratio that can be achieved, i. e. $C(\infty) = 9$.*

2.2 The Maximum Reach for a Given Competitive Ratio

Now we want to answer the following question: Given a competitive ratio $C < 9$, what is the maximum reach $r(C)$ a strategy of ratio C can achieve?

Let $S = (f_1, f_2, \dots)$ be defined by the property that in (0) and (1) equality holds, so that

$$2f_1 + 1 = C \quad \text{and} \quad f_{n+1} = \frac{C-3}{2}f_n - \sum_{i=1}^{n-1} f_i. \quad (7)$$

Then S is competitive with ratio C . By (3) it follows that

$$f_{n+1} = a_{n-1}f_1 \quad (8)$$

for all $n \geq 1$; observe that the coefficients a_i, b_i only depend on C , and not on the numbers f_i that describe the strategy.

Again, let s be the first index such that $a_s < 0$, or equivalently, $f_{s+2} < 0$ holds.

Theorem 2 *The strategy S defined by (7) has reach f_s , and this is optimal for the given ratio C .*

PROOF. From (7) we obtain by subtraction

$$f_{n+1} - f_n = \frac{C-3}{2}(f_n - f_{n-1}) - f_{n-1},$$

hence

$$f_{n+1} = \frac{C-1}{2}(f_n - f_{n-1})$$

for each $n \geq 1$. Since f_{s+2} is the first negative number this implies

$$0 < f_{s+1} < f_s.$$

Now suppose that the target lies within distance f_s of the start point. If it is discovered during the exploration of depth f_s then, by construction, the total path is at most C times the distance to the target. The same is true if

the target lies on the other side of the start point within distance f_{s+1} . If its distance equals $f_{s+1} + d \in (f_{s+1}, f_s]$ then the ratio is even smaller because of

$$\frac{\text{path to target}}{\text{distance to target}} = \frac{\text{path to } f_{s+1} + d}{f_{s+1} + d} \leq \frac{\text{path to } f_{s+1}}{f_{s+1}} \leq C.$$

This proves that strategy S' has reach f_s where $S' = (f_1, f_2, \dots, f_{s-1}, f_s, f_s)$. It remains to show that no other strategy T achieving competitive ratio C can have a bigger reach. Let $T = (g_1, g_2, \dots)$ also be of ratio C . Then, by (0) and the definition of S' we have

$$g_1 \leq \frac{C-1}{2} = f_1,$$

and from (3) and (8) we obtain

$$g_n \leq a_{n-2}g_1 \leq a_{n-2}f_1 = f_n$$

for each n where $g_n > 0$. This shows that T has at most the reach of S' . \square

2.3 Monotonicity of the Reach

It is not difficult to formalize the computation of the highest reach $r(C)$ for a given factor C . In the proof of Theorem 2 we have seen that the steps of the finite strategy $S(C)$ with highest reach f_s for a given factor C may be defined by

$$f_{k+1} = \begin{cases} (C-1)/2 & : k = 0 \\ a_{k-1}(C-1)/2 & : 1 \leq k \leq s-1 \\ a_{s-2}(C-1)/2 & : k = s \end{cases}$$

and the values $a_{k-1} > 0$ for $i = 0, \dots, s$ are given by (4). So for every $3 \leq C < 9$ we use the closed form of a_n in (5) and compute the smallest index s with $a_s < 0$. Then $a_{s-2}\frac{C-1}{2}$ is the highest reach with respect to factor C .

The function $r(C)$ is strictly monotone for $C \in [3, 9)$ and tends to ∞ if C tends to 9.

To see that $r(C)$ is strictly monotone consider factors $C, C' \in [3, 9)$ with $C' > C$. We show that for C' there is always a strategy T that achieves a higher reach than the optimal strategy $S(C) = (f_1, f_2, \dots, f_{s-1}, f_s, f_s)$ and

then in turn the reach of the optimal strategy $S(C')$ is at least as high as the reach of T . To construct a simple $T = (g_1, g_2, \dots)$ just take $g_1 = f_1, g_2 = f_2, \dots, g_{s-1} = f_{s-1}$. Then obviously

$$2g_1 + 1 < C' \quad \text{and} \quad g_{n+1} < \frac{C' - 3}{2}g_n - \sum_{i=1}^{n-1} g_i \quad (9)$$

for $n = 1, \dots, s - 2$ follows from $C' > C$. For the next two steps g_s, g_{s+1} in T we demand again equality in (9). Since $C' > C$ holds we have $g_s > f_s$ and from this we can even conclude $g_{s+1} > f_{s+1}$. Now $T = (f_1, f_2, \dots, f_{s-1}, g_s, g_s)$ has higher reach than $S(C)$ and T is C' -competitive.

It is also very easy to see that $r(C)$ has to be unbounded. Even the doubling strategy of Baeza-Yates et al. [1] achieves a factor $C = 9 - \varepsilon(r) < 9$ if the goal is $r \geq 1$ steps away from the origin. So for every $r \geq 1$ there is always a finite strategy S with reach r and factor $C < 9$ and in Theorem 2 we have shown that $r(C) \geq r$ holds.

Figure 1 shows an extract of the curve of $r(C)$. Notice the kinks in the curve, for every C inside the interval between two kinks the corresponding strategy $S(C)$ spends the same number of steps for getting the highest reach, i. e. the corresponding a_s is negative for the same index s .

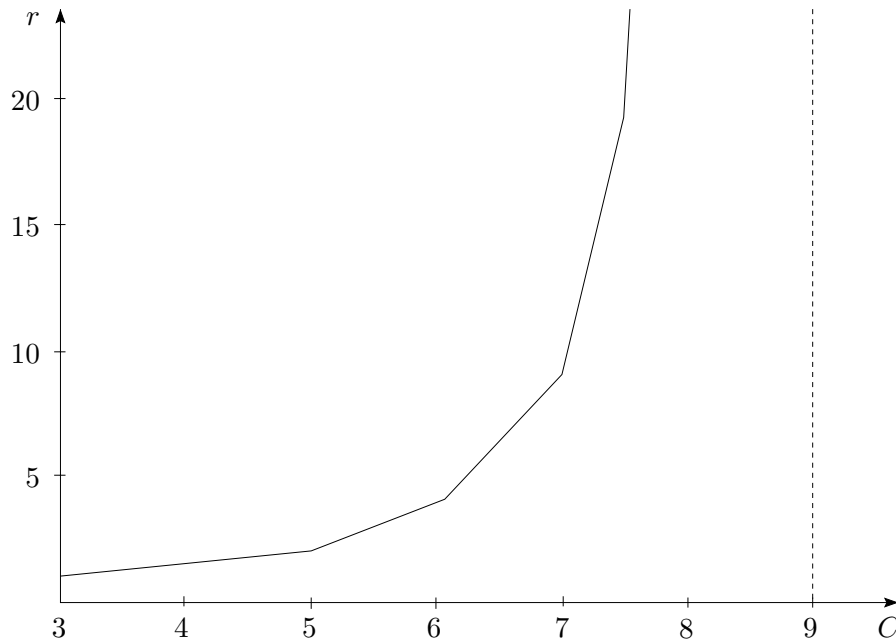


Fig. 1. Reach r versus factor C .

Now we turn back to our aim to compute the optimum competitive ratio $C(r)$ that can be achieved if the target is at least one and at most r unit steps away. Since $r(C)$ is strictly monotone we just invert $r(C)$ and obtain the ratio $C(r)$ for each length $r \in [1, \infty]$.

Corollary 3 *For $r \in [1, \infty]$ the ratio $C(r)$ is strictly monotone and takes on all values in the interval $[3, 9]$.*

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